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THE LOCAL ISOMETRIC EMBEDDING IN R^3 OF 2-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH NONNEGATIVE CURVATURE

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0. Introduction

In this paper, we will study the local isometric embedding into R^3 of 2-dimensional Riemannian manifold. Suppose that the first fundamental form $E\,du^2 + 2F\,du\,dv + G\,dv^2$ is given in a neighborhood of p. We want to find three functions $x(u,v)$, $y(u,v)$, $z(u,v)$, such that

$$dx^2 + dy^2 + dz^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{0.1}$$

in a neighborhood of p.

This embedding problem has already been solved when the Gaussian curvature K does not vanish at p. It is still an open problem when K vanishes at p. Actually, A. V. Pogorelov gave a counterexample that there exists a $C^{2,1}$ metric with no C^2 isometric embedding in R^3. In Pogorelov's example, in any neighborhood of p, there is a sequence of disjoint balls in which the metric is flat. And the Gaussian curvature K of this metric is nonnegative. The main theorem of the paper is the following.

Main Theorem. *Suppose that the Gaussian curvature of a C^s metric is nonnegative for $s \geqslant 10$, then there exists a C^{s-6} isometric embedding in R^3.*

Instead of studying the nonlinear system (0.1) of first order, we will study a second-order Monge-Ampére equation satisfied by a coordinate, say z. The equation can be derived as follows: If the Gaussian curvature of $E\,du^2 + 2F\,du\,dv + G\,dv^2 - dz^2$ vanishes, then z must satisfy

$$\begin{aligned}(z_{11} - \Gamma^i_{11} z_i)(z_{22} - \Gamma^i_{22} z_i) - (z_{12} - \Gamma^i_{12} z_i)^2 \\ = K\{EG - F^2 - Ez_2^2 - Gz_1^2 + 2Fz_1 \cdot z_2\} \equiv K(u,v,\nabla z),\end{aligned} \tag{0.2}$$

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where $z_1 = (\partial z/\partial u)$, $z_2 = (\partial z/\partial v)$, z_{ij} are second derivative of z, and Γ^i_{jk} are symbols. Conversely, suppose z satisfies (0.2), then the metric $E\,du^2 + 2F\,du\,dv + G\,dv^2 - dz^2$ is flat. Hence there exists a coordinate system x, y, such that $dx^2 + dy^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 - dz^2$ which is (0.1).

In this paper, we will prove that there exists a smooth local solution of (0.2), provided K is nonnegative.

We may assume p is the origin $(0,0)$, and $K(0,0,0) = 0$. Set $u = \varepsilon^2 x$, $v = \varepsilon^2 y$, $z = (v^2/2) + \varepsilon^5 w$. (0.2) becomes

$$\left(\varepsilon w_{xx} - \varepsilon^2\Gamma^2_{11}y - \varepsilon^3\Gamma^l_{11}w_{x_l}\right)\left(1 + \varepsilon w_{yy} - \varepsilon^2\Gamma^2_{22}y - \varepsilon^3\Gamma^l_{22}w_{x_l}\right)$$
$$-\left(\varepsilon w_{xy} - \varepsilon^2\Gamma^2_{12}y - \varepsilon^3\Gamma^l_{12}w_{x_l}\right)^2 - K\left(\varepsilon^2 x, \varepsilon^2 y, \varepsilon^3\nabla w\right) = 0,$$

where $x_1 = x$, $x_2 = y$. Cancelling ε on both sides, we have

$$w_{xx} + \varepsilon\tilde{F}\left(\varepsilon, x, y, \nabla w, \nabla^2 w\right) = 0, \tag{0.3}$$

where

$$\tilde{F}\left(\varepsilon, x, y, \nabla w, \nabla^2 w\right) = \left(w_{xx} - \varepsilon\Gamma^2_{11}y - \varepsilon^2\Gamma^l_{11}w_{x_l}\right)\left(w_{yy} - \varepsilon\Gamma^2_{22}y - \varepsilon^2\Gamma^l_{22}w_{x_l}\right)$$
$$-\left(w_{xy} - \varepsilon\Gamma^2_{12}y - \varepsilon^2\Gamma^l_{12}w_{x_l}\right)^2 - \Gamma^2_{11}y - \varepsilon\Gamma^l_{11}w_{x_l} - \left(K\left(\varepsilon^2 x, \varepsilon^2 y, \varepsilon^3\nabla w\right)\right)/\varepsilon^2.$$

Fix $x_0, y_0 > 0$, consider a rectangle D: $D = \{(x,y) \mid |x| \leqslant x_0,\ |y| \leqslant y_0\}$. Choose two nonnegative cut-off function $\chi_i \in C^\infty(D)$ as follows:

$$\chi_1 = \begin{cases} 1 & \text{if } |Y| \leqslant \dfrac{y_0}{2}, \\ 0 & \text{if } |y| \geqslant \dfrac{3y_0}{4}, \end{cases} \qquad \chi_2 = \begin{cases} 1 & \text{if } |y| \leqslant \dfrac{3y_0}{4}, \\ 0 & \text{if } |y| \geqslant \dfrac{7y_0}{8}; \end{cases}$$

cut-off the nonlinear term by

$$F\left(\varepsilon, x, y, \nabla w, \nabla^2 w\right)$$
$$= \chi_1\left\{\left(w_{xx} - \varepsilon\Gamma^2_{11}y - \varepsilon^2\Gamma^l_{11}w_{x_l}\right)\left(w_{yy} - \varepsilon\Gamma^2_{22}y - \varepsilon^2\Gamma^l_{22}w_{x_l}\right)\right.$$
$$\left. -\left(w_{12} - \varepsilon\Gamma^2_{12}y - \varepsilon^2\Gamma^l_{12}w_{x_l}\right)^2 - \frac{K\left(\varepsilon^2 x, \varepsilon^2 y, \varepsilon^3\nabla w\right)}{\varepsilon^2}\right\}$$
$$-\varepsilon\chi_2\left(\Gamma^l_{11}w_{x_l} - \Gamma^2_{11}y\right).$$

In the following, we will consider the following equation instead of (0.3):

$$w_{xx} + \varepsilon F\left(\varepsilon, x, y, \nabla w, \nabla^2 w\right) = 0. \tag{0.4}$$

For any smooth function w defined in D, define

$$G(w) = w_{xx} + \varepsilon F\left(\varepsilon, x, y, \nabla w, \nabla^2 w\right). \tag{0.5}$$

Lemma 0.1. *Suppose $|w|_{C^2(D)} \leqslant 1$, and $\theta > 0$ be a constant such that*

$$|G(w)|_{L^\infty(D)} \leqslant \theta.$$

Then if ε is sufficiently small, $L_\theta(w)\rho = L(w)\rho + \theta\chi_1\rho_{yy}$ is a degenerate elliptic second-order equation where $L(w)\rho$ is the linearized equation of (0.4) *about w.*

Proof. Suppose the linearized equation is $L(w)\rho = \rho_{xx} + \varepsilon\sum a_{ij}\rho_{x_i x_j} +$ lower order term. We want to prove the determinant of

$$A = \begin{pmatrix} 1 + \varepsilon a_{11} & \varepsilon a_{12} \\ \varepsilon a_{12} & \varepsilon a_{22} + \theta\chi_1 \end{pmatrix}$$

is nonnegative. The determinant is, after a straight computation,

$$\varepsilon a_{22}(1 + \varepsilon a_{11}) - \varepsilon^2 a_{12}^2 + \theta\chi_1(1 + \varepsilon a_{11}) = \varepsilon\chi_1 G(w) + \chi_1^2 K + \theta\chi_1(1 + \varepsilon a_{11}).$$

In the computation, we use $\chi_1 \cdot \chi_2 = \chi_1$. So if ε is small, then the determinant $\geqslant 0$. q.e.d.

In the following sections, we will prove that there exists a smooth solution of (0.4). In §1, we will study existence, regularity, and estimates of the degenerate elliptic equation $L_\theta(w)$. In §2, we will modify the Nash-Moser-Hörmander's iterative scheme to solve (0.4). Then we will complete the proof of the Main Theorem.

1. Linear theory

In this section L will represent as a degenerate elliptic operator of second-order defined in a rectangle $D = \{(x, y) \mid |x| \leqslant x_0,\ |y| \leqslant y_0\}$. Consider the following boundary value problem:

$$\text{(1.1)} \qquad \begin{aligned} &L\rho = \rho_{xx} + \sum_{i,j=1}^{2} a_{ij}\rho_{x_i x_j} + a_1\rho_x + a_2\rho_y + a\rho = g \quad \text{in } D; \\ &\rho(x_0, y) = \rho(-x_0, y) = 0. \end{aligned}$$

Assumption. All the coefficients a_{ij}, a_i, and a vanish near $y = \pm y_0$. And $\sum|a_{ij}|_{C^4} + |a_i|_{C^4} + |a|_{C^4} \leqslant C_0\varepsilon$, where C_0 is a fixed constant.

Set

$$\rho(x, y) = u(x, y)e^{-\lambda x^2}, \qquad \lambda > 0.$$

Then (1.1) becomes

$$\text{(1.2)} \qquad \begin{aligned} &Lu = u_{xx} + \sum_{i,j=1}^{2} a_{ij}u_{x_i x_j} + \sum_{i=1}^{2} b_i u_{x_i} + hu = e^{\lambda x^2}g, \\ &u(x_0, y) = u(-x_0, y) = 0, \end{aligned}$$

where

$$
(1.3)\qquad
\begin{aligned}
b_1 &= -4(1 + a_{11})\lambda x + a_1,\\
b_2 &= -4a_{12}\lambda x + a_2,\\
h &= -2(1 + a_{11})\lambda + 4(1 + a_{11})\lambda^2 x^2 - 2b_1\lambda x + a.
\end{aligned}
$$

Instead of studying equation (1.2), we will consider the following regularization of (1.2):

$$
(1.4)\qquad
\begin{aligned}
&L_\nu u = -\nu\left[D^*D - \frac{\partial^2}{\partial x^2}\right]u + Lu = g \quad \text{in } D;\\
&u(x_0, y) = u(-x_0, y) = 0;
\end{aligned}
$$

where $Du = (y_0^2 - y^2)(\partial u/\partial y)$, D^* is the adjoint of D, and $\nu > 0$ is a small constant. λ will be chosen large but independent of ν and ε, and always satisfies $\lambda x_0 < 1$.

Theorem 1.1. *Suppose all coefficients are smooth and ε, ν are small. Then there exists $s_0(\varepsilon, \nu) > 0$ such that for any $g \in H^s(D)$, $s \leqslant s_0$, there exists a unique solution $u \in H^s(D)$ of* (1.4) *and the following estimates are true*:

$$
(1.5)\qquad \|u\|_{H^s} \leqslant C_s\{\|g\|_{H^s} + \Gamma(s)\|u\|_{H^2}\},
$$

where

$$
\Gamma(s) = \sum_{i,j}\left\{\|a_{ij}\|_{H^{s+2}} + \|b_i\|_{H^{s+1}} + \|h\|_{H^s}\right\},
$$

and C_s is a constant which is independent of ν and ε.

H^s is the Sobolev space with the norm: $\|u\|_{H^s} = (\sum_{|\alpha| \leqslant s}\|D^\alpha u\|_{L^2}^2)^{1/2}$ where D^α is any αth derivative.

Throughout the section, C always be a constant which is independent of ν, and will change from line to line. $\lambda > 0$ will be a fixed number throughout. We will divide the proof of Theorem 1.1 into several lemmas. First, we will prove the existence of weak solution of (1.4).

Suppose u, ϕ are smooth functions and satisfy the boundary conditions $u(x_0, y) = u(-x_0, y) = \phi(x_0, y) = \phi(-x_0, y) = 0$. Then

$$
\begin{aligned}
Q_\nu(\phi, u) \equiv -(\phi, L_\nu u) = \nu\left[\int \phi_x u_x + \int D\phi Du\right] &+ \int \phi_x u_x + \int a_{ij} u_{x_i}\phi_{x_j}\\
&+ \frac{1}{2}\sum_{i=1}^{2}\int\left(b_i - \sum_{j=1}^{2}\frac{\partial a_{ij}}{\partial x_j}\right)(\phi_{x_i}u - u_{x_i}\phi)\\
&+ \int\left[-h + \frac{1}{2}\left(\sum_{i=1}^{2}\frac{\partial b_i}{\partial x_i} - \sum_{i,j=1}^{2}\frac{\partial a_{ij}}{\partial x_i\partial x_j}\right)\right]\phi u.
\end{aligned}
$$

Define $\mathring{H}^1$ as the space consisted of functions u such that u, Du, u_x are in $L^2(D)$, and satisfy $u(x_0, y) = u(-x_0, y) = 0$,

$$|||u||| = \|u\|_{L^2} + \|Du\|_{L^2} + \|u_x\|_{L^2}.$$

Lemma 1.2 (*existence of weak solution*). *Given $g \in L^2(D)$, then there exists a unique $u \in \mathring{H}^1$ such that*

$$Q_\nu(\phi, u) = -(\phi, g) \quad \textit{for any } \phi \in \mathring{H}^1.$$

Proof. $Q_\nu(\phi, u)$ is a bounded bilinear form of $\mathring{H}^1$. We want to prove

$$Q_\nu(\phi, \phi) \geqslant C_\nu |||\phi|||^2 \quad \forall \phi \in \mathring{H}^1. \tag{1.6}$$

Because $\partial b_1/\partial x$ involves λ, we write

$$Q_\nu(\phi, \phi) = \nu\left[\int \phi_x^2 + \int |D\phi|^2\right] + \int \phi_x^2 + \sum_{i,j=1}^{2} \int a_{ij} \frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_j}$$
$$- \sum_{i=1}^{2} \int b_i \phi_{x_i} \phi + \int \left[-h + \sum_{i,j}\left(\frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}\right)\right]\phi^2.$$

We note

$$\int b_2 \phi_y \phi = \frac{1}{2}\int b_2 \frac{\partial \phi^2}{\partial y} = -\frac{1}{2}\int \frac{\partial b_2}{\partial y}\phi^2, \quad \text{so that} \left|\int b_2 \phi_y \phi\right| \leqslant C\varepsilon \int \phi^2.$$

Thus we only have to estimate $\int b_1 \phi_x \phi$. Suppose $\lambda_1 > \lambda_2$ be eigenvalues of

$$\begin{pmatrix} 1 + a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix},$$

and v^1, v^2 are unit eigenvectors such that

$$v^1 = \begin{pmatrix} v_1^1 \\ v_2^1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

near $y = \pm y_0$. Define ϕ_1, ϕ_2 by the following

$$\begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix} = \phi_1 v^1 + \phi_2 v^2, \tag{1.7}$$

since v^2 is the eigenvector with eigenvalue λ_2,

$$v_1^2 = -\frac{a_{12} v_2^2}{1 + a_{11} - \lambda_2}$$

is small. Also by the relation of u_x, u_y, u_1, u_2, we have

$$\phi_x = \frac{v_1^1 \phi_1 + v_1^2 v_2^2 \phi_y}{1 - (v_1^2)^2}, \tag{1.7$'$}$$

and therefore

$$\int b_1\phi_x\phi = \frac{1}{2}\int \frac{b_1 v_1^2 v_2^2(\phi^2)y}{1-(v_1^2)^2} + \int \frac{b_1 v_1^1\phi_1\phi}{1-(v_1^2)^2}$$

$$= -\frac{1}{2}\int \left(\frac{\partial}{\partial y}\frac{b_1 v_1^2 v_2^2}{1-(v_1^2)^2}\right)\phi^2 \int \frac{b_1 v_1^1\phi_1\phi}{1-(v_k^2 1)^2}.$$

Hence

$$\left|\int b_1\phi_x\phi\right| \leqslant C\int \phi^2 + \frac{1}{2}\int \lambda_1\phi_1^2,$$

where we use Schwartz inequality and the fact that λ_1 is close to 1 when ε is small, and C is a constant independent of λ. Here $\lambda x_0 < 1$ is required. Hence, if $\lambda - C \geqslant 1$, then

$$\begin{aligned} Q_\nu(\phi,\phi) &\geqslant \nu\left[\int \phi_x^2 + \int |D\phi|^2\right] \\ &\quad + \frac{1}{2}\int \lambda_1\phi_1^2 + \int \lambda_2\phi_2^2 + (\lambda - C)\int \phi^2 \geqslant \nu|||\phi|||^2. \end{aligned} \tag{1.8}$$

Then we apply Lax-Milgram's theorem to get a weak solution. q.e.d.

We will prove that the weak solution is smooth provided g is smooth. Since L_ν is elliptic inside D, u may be supposed smooth inside D by regularity theorem of elliptic equation. We only have to prove that u is smooth up to boundary of D.

Lemma 1.3. *Suppose* $g \in H^s(D)$, $\nu s^2 < 1$, λ *is large, and* u *is the weak solution of* (1.4), *then* $u, Du, u_x \in H^s(D)$.

Proof. Define $a_{\bar\varepsilon}(y) \geqslant 0$ as follows:

$$a_{\bar\varepsilon}(y) = \begin{cases} y_0^2 - y^2 & \text{if } -y_0 + \bar\varepsilon \leqslant y \leqslant y_0 - \bar\varepsilon, \\ 0 & \text{if } y \geqslant y_0 - \dfrac{\bar\varepsilon}{2},\ y \leqslant -y_0 + \dfrac{\bar\varepsilon}{2}, \end{cases} \tag{1.9}$$

$$\left|\frac{\partial a_{\bar\varepsilon}(y)}{\partial y}\right| \leqslant C_1,$$

where C_1 is a constant independent of $\bar\varepsilon$. Define $D_{\bar\varepsilon}u = a_{\bar\varepsilon}(\partial u/\partial y)$. Differentiating (1.4) by $D_{\bar\varepsilon}$, we have $L_\nu D_{\bar\varepsilon}u = D_{\bar\varepsilon}g + [L_\nu, D_{\bar\varepsilon}]u$. Taking the inner product with $D_{\bar\varepsilon}u$, we have

$$\begin{aligned} \nu\left[\int |D_{\bar\varepsilon}u_x|^2 + |DD_{\bar\varepsilon}u|^2\right] &\leqslant -(D_{\bar\varepsilon}u, L_\nu D_{\bar\varepsilon}u) \\ &= -(D_{\bar\varepsilon}u, D_{\bar\varepsilon}h) - (D_{\bar\varepsilon}u, [L_\nu, D_{\bar\varepsilon}]u). \end{aligned}$$

Since we have already known $u \in \mathring{H}^1$,

$$|(D_{\bar\varepsilon}u, D_{\bar\varepsilon}g)| \leqslant \|D_{\bar\varepsilon}u\|_{L^2} \cdot \|D_{\bar\varepsilon}g\|_{L^2} \leqslant C_2,$$

where C_2 is independent of $\bar\varepsilon$. Thus

$$\begin{aligned}[L_\nu, D_{\bar\varepsilon}] &= -\nu[D^*D, D_{\bar\varepsilon}] + [L, D_{\bar\varepsilon}] \\ &= -\nu\left(D^*[D, D_{\bar\varepsilon}] + [D^*, D_{\bar\varepsilon}]D\right) + [L, D_{\bar\varepsilon}],\end{aligned}$$

$$\begin{aligned}\left|\left(D_{\bar\varepsilon}u, D^*[D, D_{\bar\varepsilon}]u\right)\right| &= \left|\left(DD_{\bar\varepsilon}u, [D, D_{\bar\varepsilon}]u\right)\right| \\ &\leqslant C\|Du\|_{L^2} \cdot \|DD_{\bar\varepsilon}u\|_{L^2} \leqslant C_2\|DD_{\bar\varepsilon}u\|_{L^2},\end{aligned}$$

by (1.9). Similarly,

$$\left\|\left(D_{\bar\varepsilon}u, [D^*, D_{\bar\varepsilon}]Du\right)\right\| \leqslant C_3\|DD_{\bar\varepsilon}u\|_{L^2}.$$

Because each term in $[L, D_{\bar\varepsilon}]$ involves a_{ij}, b_i, and y-derivatives of a_{ij}, b_j which vanish near $y = \pm y_0$, $[L, D_{\bar\varepsilon}] = [L, D]$ for $\bar\varepsilon$ is small. Combining all estimates, gives

$$\nu\left(\int |D_{\bar\varepsilon}u_x|^2 + \int |DD_{\bar\varepsilon}|^2\right) \leqslant C_4\|DD_{\bar\varepsilon}u\|_{L^2},$$

so that

$$\int |D_{\bar\varepsilon}u_x|^2 + \int |DD_{\bar\varepsilon}u|^2 \leqslant C_5(\nu),$$

where C_5 is a constant independent of $\bar\varepsilon$. Taking the limit $\bar\varepsilon \to 0$, we have

$$\int |Du_x|^2 + \int |D^2u|^2 < +\infty.$$

From (1.4) we also conclude

$$u_{xx} \in L^2(D).$$

Define

$$\tilde a_{\bar\varepsilon}(y) = \begin{cases} 1 & \text{if } y^2 \leqslant y_0^2 - \bar\varepsilon^2, \\ 0 & \text{if } y^2 = y_0^2, \\ \text{linear in between.} \end{cases}$$

Define $\tilde D_{\bar\varepsilon}u = \tilde a_{\bar\varepsilon}(y)(\partial u/\partial y)$, and $Du = (y_0^2 - y^2)(\partial u/\partial y) \equiv a(y)(\partial u/\partial y)$. By the previous step, we know

$$D\tilde D_{\bar\varepsilon}u,\ \tilde D_{\bar\varepsilon}u_x \in L^2(D).$$

Differentiating (1.4) by $\tilde D_{\bar\varepsilon}$, we have

$$L_\nu(\tilde D_{\bar\varepsilon}u) = \tilde D_{\bar\varepsilon}g + [L_\nu, \tilde D_{\bar\varepsilon}]u.$$

Taking the inner product with $\tilde{D}_{\bar{\varepsilon}}u$, yields

$$\nu\left[\int |D\tilde{D}_{\bar{\varepsilon}}u|^2 + |\tilde{D}_{\bar{\varepsilon}}u_x|^2\right] + (\lambda - C)\|\tilde{D}_{\bar{\varepsilon}}u\|_{L^2}^2 \leqslant -(\tilde{D}_{\bar{\varepsilon}}u, \tilde{D}_{\bar{\varepsilon}}g) + (\tilde{D}_{\bar{\varepsilon}}u, [\tilde{D}_{\bar{\varepsilon}}, L_\nu]u),$$

by (1.8).

$$[L_\nu, \tilde{D}_{\bar{\varepsilon}}] = -\nu[D^*, \tilde{D}_{\bar{\varepsilon}}]D - \nu D^*[D, \tilde{D}_{\bar{\varepsilon}}] + [L, \tilde{D}_{\bar{\varepsilon}}],$$

$$[D, \tilde{D}_{\bar{\varepsilon}}] = \left(a\frac{\partial a_{\bar{\varepsilon}}}{\partial y} - a_{\bar{\varepsilon}}\frac{\partial a}{\partial y}\right)\frac{\partial}{\partial y}.$$

Since $(\partial\tilde{a}_{\bar{\varepsilon}}/\partial y) = \text{const} \neq 0$ only for $y_0^2 - \bar{\varepsilon}^2 \leqslant y^2 \leqslant y_0^2$, $|[D, \tilde{D}_{\bar{\varepsilon}}]u| \leqslant C_1|\tilde{D}_{\bar{\varepsilon}}u|$, for some constant C_1 independent of ε. Thus

$$\left|\left(\tilde{D}_{\bar{\varepsilon}}u, D^*[D, \tilde{D}_{\bar{\varepsilon}}]u\right)\right| \leqslant C_2\|D\tilde{D}_{\bar{\varepsilon}}u\|_{L^2} \cdot \|\tilde{D}_{\bar{\varepsilon}}u\|_{L^2}.$$

Similarly, we have

$$\left|\left(\tilde{D}_{\bar{\varepsilon}}u, [D^*, \tilde{D}_{\bar{\varepsilon}}]Du\right)\right| \leqslant C_3\left(\|D\tilde{D}_{\bar{\varepsilon}}u\|_{L^2} + 1\right)\|D_{\bar{\varepsilon}}u\|_{L^2}.$$

As before, $[L, D_{\bar{\varepsilon}}]$ is independent of $\bar{\varepsilon}$ if $\bar{\varepsilon}$ is small. Hence, we have

$$\begin{aligned}(1.10)\quad &\nu\left[\|D\tilde{D}_{\bar{\varepsilon}}u\|_{L^2}^2 + \|\tilde{D}_{\bar{\varepsilon}}u_x\|_{L^2}^2\right] + (\lambda - C)\|\tilde{D}_{\bar{\varepsilon}}u\|_{L^2}^2 \\ &\leqslant C_4\left\{\|\tilde{D}_{\bar{\varepsilon}}u\|_{L^2} \cdot \|g\|_{H^1} + \nu\|D\tilde{D}_{\bar{\varepsilon}}u\|_{L^2} \cdot \|\tilde{D}_{\bar{\varepsilon}}u\|_{L^2} + \|\tilde{D}_{\bar{\varepsilon}}u\|_{L^2}\right\}\end{aligned}$$

for a constant C_4 independent of $\bar{\varepsilon}$. Using Schwartz inequality, we have

$$\|D\tilde{D}_{\bar{\varepsilon}}u\|_{L^2} + \|\tilde{D}_{\bar{\varepsilon}}u_x\|_{L^2} + \|\tilde{D}_{\bar{\varepsilon}}u\|_{L^2} \leqslant C_5$$

independent of $\bar{\varepsilon}$. Taking the limit $\bar{\varepsilon}\downarrow 0$, we have $u_y, Du_y, u_{yx} \in L^2$, which is the case $s = 1$.

We can prove Lemma 1.3 by induction on s. Now suppose $u, u_x, Du \in H^s$; we want to prove $u, u_x, Du \in H^{s+1}$. Differentiating (1.4) by $\partial^s/\partial y^s$, we have

$$\begin{aligned}(1.11)\quad &L_\nu\left(\frac{\partial^s}{\partial y^s}u\right) + 2s\nu\frac{\partial a}{\partial y}D\left(\frac{\partial^s u}{\partial y^s}\right) + \frac{s(s-1)}{2}\nu\frac{\partial^2 a^2(y)}{\partial y^2}\frac{\partial^s u}{\partial y^s} \\ &= \frac{\partial^s g}{\partial y^s} + \text{other term} \equiv \tilde{g}_s.\end{aligned}$$

The other term in the above expression consists of derivatives of order $s + 1$, or s with vanishing coefficients near $y = \pm y_0$, and derivative of order $< s$; hence $\tilde{g}_s \in H^1(D)$. As the same proof in the previous step, it is easy to prove

$$D^2\left(\frac{\partial^s u}{\partial y^s}\right) \quad \text{and} \quad D\left(\frac{\partial^s u_x}{\partial y^s}\right) \quad \text{in } L^2(D).$$

Differentiating (1.11) by $D_{\bar{\varepsilon}}$ and doing the same steps as (1.10), we have

$$\nu\left[\left\|D\tilde{D}_{\bar{\varepsilon}}\frac{\partial^s u}{\partial y^s}\right\|_{L^2}^2+\left\|\tilde{D}_{\bar{\varepsilon}}\frac{\partial^{s+1}u}{\partial y^s\partial x}\right\|_{L^2}^2\right]+(\lambda-C)\left\|\tilde{D}_{\bar{\varepsilon}}\frac{\partial^s u}{\partial y^s}\right\|_{L^2}^2$$

$$\leqslant C_4\left\{\|\tilde{g}_s\|_{H^1}\cdot\left\|\tilde{D}_{\bar{\varepsilon}}\frac{\partial^s u}{\partial y^s}\right\|_{L^2}+\nu\left\|D\tilde{D}_{\bar{\varepsilon}}\frac{\partial^s u}{\partial y^s}\right\|_{L^2}\cdot\left\|\overline{D}_{\bar{\varepsilon}}\frac{\partial^s u}{\partial y^s}\right\|_{L^2}\right.$$

$$\left.+\left\|\overline{D}_{\bar{\varepsilon}}\frac{\partial^s u}{\partial y^s}\right\|_{l^2}+\nu s^2\left\|\overline{D}_{\bar{\varepsilon}}\frac{\partial^s u}{\partial y^s}\right\|_{L^2}^2\right\}.$$

Hence if $\nu s^2<1$ and λ is large, then

$$\left\|D\tilde{D}_{\bar{\varepsilon}}\frac{\partial^s u}{\partial y^s}\right\|_{L^2}+\left\|\tilde{D}_{\bar{\varepsilon}}\frac{\partial^{s+1}u}{\partial y^s\partial x}\right\|_{L^2}+\left\|\tilde{D}_{\bar{\varepsilon}}\frac{\partial^s u}{\partial y^s}\right\|_{L^2}$$

is bounded and independent of $\bar{\varepsilon}$. Taking the limit $\bar{\varepsilon}\downarrow 0$, we prove

$$D\frac{\partial^{s+1}u}{\partial y^{S+1}},\ \frac{\partial^{s+2}u}{\partial y^{s+d1}x},\ \frac{\partial^{s+1}u}{\partial y^{s+1}}\in L^2(D).$$

From (1.4), $(1+\nu+a_{11})u_{xx}=g+D^*Du+$ terms with coefficients vanishing on $y=\pm y_0+$ lower order terms. Differentiating the above express by $(\partial^k/\partial x^k)(\partial^{s-k}/\partial y^{s-k})$, $k=0,1,2,\cdots,s$, we conclude with $u_x\in H^{s+1}(D)$. The fact

$$\frac{\partial^s u}{\partial y^s},\ D\frac{\partial^s u}{\partial y^s}\in H^1(D)\quad\text{and}\quad u_x\in H^{s+1}(D)$$

implies u, Du and $u_x\in H^{s+1}(D)$. Thus we have finished the induction step.

Proof of Theorem 1.1. To prove estimate (1.5), we may assume g and u are both smooth functions. Differentiating (1.4) by $\partial^s/\partial y^s$, we have

$$L_\nu\left(\frac{\partial^s u}{\partial y^s}\right)=\frac{\partial^s g}{\partial y^s}+\left[L_\nu,\frac{\partial^s}{\partial y^s}\right]u.$$

We want to estimate

$$\left(\frac{\partial^s u}{\partial y^s},\left[L_\nu,\frac{\partial^s}{\partial y^s}\right]u\right).$$

Since

$$\left[D^*D,\frac{\partial^s}{\partial y^s}\right]u=-2sD^*\frac{\partial a}{\partial y}\frac{\partial^s u}{\partial y^s}+\text{ terms with } y \text{ derivatives of order }\leqslant s,$$

$$\left|\left(\frac{\partial^s u}{\partial y^s},D^*\frac{\partial a}{\partial y}\frac{\partial^s u}{\partial y^s}\right)\right|=\left|\frac{1}{2}\int\frac{\partial a}{\partial y}D\left(\frac{\partial^s u}{\partial y^s}\right)^2\right|$$

$$=\frac{1}{2}\left|\int D^*\frac{\partial a}{\partial y}\left(\frac{\partial^s u}{\partial y^s}\right)^2\right|\leqslant C\int\left|\frac{\partial^s u}{\partial y^s}\right|^2,$$

we have

$$\left|\left(\frac{\partial^s u}{\partial y^s}, \left[D^*D, \frac{\partial^s}{\partial y^s}\right]u\right)\right| \leqslant \sum_{l \leqslant s} C_s \left\|\frac{\partial^l u}{\partial y^l}\right\|_{L^2}^2. \tag{1.12}$$

Let D^α denote any derivative of order $|\alpha|$. We will use following inequalities which come from interpolational inequalities immediately:

$$\|D^\alpha u D^\beta v\|_{L^2} \leqslant C_s(\|u\|_{L^\infty}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{L^\infty}), \tag{1.13}$$

where $|\alpha| + |\beta| = s$, and $\|u\|_{L^\infty} \leqslant C\|u\|_{H^2}$. Using integration by part and (1.13), we can estimate

$$\begin{aligned}&\left|\left(\frac{\partial^s u}{\partial y^s}, \left[a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}, \frac{\partial^s}{\partial y^s}\right]u\right)\right| \\ &\qquad \leqslant C_s\{\|a_{ij}\|_{C^2}\|u\|_{H^s} + \|u\|_{H^2}\cdot\|a_{ij}\|_{H^{s+2}}\}\left\|\frac{\partial^s u}{\partial y^s}\right\|_{L^2}.\end{aligned} \tag{1.14}$$

Similarly,

$$\left|\left(\frac{\partial^s u}{\partial y^s}, \left[b_i\frac{\partial}{\partial x_i}, \frac{\partial^s}{\partial y^s}\right]u\right)\right| \leqslant C_s\{\|b_i\|_{C^1}\|u\|_{H^s} + \|u\|_{H^2}\|b_i\|_{H^{s+1}}\}\left\|\frac{\partial^s u}{\partial y^s}\right\|_{L^2},$$

$$\left|\left(\frac{\partial^s u}{\partial y^s}, \left[h, \frac{\partial^s}{\partial y^s}\right]u\right)\right| \leqslant C_s\{\|u\|_{H^{s-1}}|h|_{C^1} + \|h\|_{H^s}\|u\|_{H^2}\}\left\|\frac{\partial^s u}{\partial y^s}\right\|_{L^2}. \tag{1.15}$$

Combining (1.12), (1.14), (1.15) and the assumption, we have

$$\left|\left(\frac{\partial^s u}{\partial y^s}, \left[L_\nu, \frac{\partial^s}{\partial y^s}\right]u\right)\right| \leqslant C_s\{(\nu + \varepsilon)\|u\|_{H^s} + \Gamma(s)\|u\|_{H^2}\}\left\|\frac{\partial^s u}{\partial y^s}\right\|_{L^2},$$

where

$$\Gamma(s) = \sum \|a_{ij}\|_{H^{s+2}} + \|b_k\|_{H^{s+1}} + \|h\|_{H^s}.$$

Now denote $u^s = \partial^s u/\partial y^s$, and u_1^s as defined in (1.7), i.e.,

$$\begin{pmatrix} u_x^s \\ u_y^s \end{pmatrix} = u_1^s v^1 + u_2^s v^2.$$

We want to prove the following inequalities by induction on s:

$$\begin{aligned}&\nu\left(\int |u_x^s|^2 + \int |Du^s|^2\right) + \int \lambda_1(u_1^s) + \int \lambda_2(u_2^s) + \int |u^s|^2 \\ &\qquad \leqslant C_s\{\|g\|_{H^s} + \Gamma(s)\|u\|_{H^2}\}.\end{aligned} \tag{1.16$_s$}$$

(1.16)0 is just (1.8). Suppose we have proved (1.16)s–1. By (1.8),

$$\begin{aligned}&\nu\left(\int |u_x^s|^2 + |Du^s|^2\right) + \frac{1}{2}\int \lambda_1(u_1^s)^2 + \int \lambda_2(u_2^s)^2 + (\lambda - C)\int |u^s|^2 \\ &\qquad \leqslant -(u^s, L_\nu u^s) \leqslant \|u^s\|_{L^2}\cdot\|g\|_{H^s} \\ &\qquad\quad + C_s\{(\nu + \varepsilon)\|u\|_{H^s} + \Gamma(s)\|u\|_{H^2}\}\|u^s\|_{L^2}.\end{aligned} \tag{1.17$_s$}$$

By (1.7)′ we have

$$(1.18)s \qquad \begin{aligned} \|u_x^{s-1}\|_{L^2} &\leqslant C_1\left\{\|u_1^{(s-1)}\|_{L^2} + \|a_{ij}\|_{L^\infty}\|u^s\|_{L^2}\right\} \\ &\leqslant C_2\left\{\|u_1^{(s-1)}\|_{L^2} + \varepsilon\|u^s\|_{L^2}\right\}. \end{aligned}$$

Solving for u_{xx} from (1.4), and differentiating by

$$\frac{\partial^{s-2}}{\partial x^k \partial y^{s-k-2}}, \qquad k = 0, \cdots, s-2,$$

we have

$$\left\|\frac{\partial^s u}{\partial x^{k+2}\partial y^{s-k-2}}\right\|_{L^2} \leqslant C_k\left\{\left\|\frac{\partial^s u}{\partial x^{k+1}\partial y^{s-k}}\right\|_{L^2} + \left\|\frac{\partial^s u}{\partial x^k \partial y^{s-k}}\right\|_{L^2} + \|g\|_{H^s} + \|u\|_{H^{s-1}} + \|u\|_{H^2}\Gamma(s)\right\}.$$

Summing over k,

$$\|u\|_{H^s} \leqslant C_s\left\{\|g\|_{H^s} + \left\|\frac{\partial^s u}{\partial y^s}\right\|_{L^2} + \left\|\frac{\partial^s u}{\partial y^{s-1}\partial x}\right\|_{L^2} + \|u\|_{H^{s-1}} + \|u\|_{H^2}\Gamma(s)\right\}.$$

By (1.18)s and (1.16)s–1,

$$(1.19)s \qquad \begin{aligned} \|u\|_{H^s} &\leqslant C_s\left\{\left\|\frac{\partial^s u}{\partial y^s}\right\|_{L^2} + \|u_1^{s-1}\|_{L^2} + \|u\|_{H^{s-1}} + \|u\|_{H^2}\Gamma(s) + \|g\|_{H^s}\right\} \\ &\leqslant C_s\left\{\left\|\frac{\partial^s u}{\partial y^s}\right\|_{L^2} + \|g\|_{H^s} + \|u\|_{H^{s-1}} + \|u\|_{H^2}\Gamma(s)\right\}. \end{aligned}$$

Now we can estimate the right-hand side of (1.17)s,

$$\nu\left(\int |u_x^s|^2 + \int |Du^s|^2\right) + \frac{1}{2}\int \lambda_1(u_1^s) + \int \lambda_2|u_2^s|^2 + (\lambda - C)\int |u^s|^2 \leqslant C_s'\left\{(\nu+\varepsilon)\left\|\frac{\partial^s u}{\partial y^s}\right\|_{L^2} + \|g\|_{H^s} + \|u\|_{H^{s-1}} + \|u\|_{H^2}\Gamma(s)\right\}\left\|\frac{\partial^s u}{\partial y^s}\right\|_{L^2}.$$

If $C_s'(\nu+\varepsilon) < 1$ and $\lambda - C \geqslant 2$, we have

$$\nu\left(\int |u_x^s|^2 + \int |Du^s|^2\right) + \frac{1}{2}\int \lambda_1(u_1^s)^2 + \int \lambda_2|u_2^s|^2 + \int |u^s|^2 \leqslant C_s'\left\{\|g\|_{H^s} + \|u\|_{H^{s-1}} + \|u\|_{H^2}\Gamma(s)\right\}\|u^s\|_{L^2}.$$

Because of (1.19)s, this is equivalent to (1.16)s, and we finish the induction proof. Then (1.5) follows from (1.16)s.

Hence we have proved Theorem 1.1. q.e.d.

Let $L(w)$ be the linearized equation of $G(w) = w_{xx} + \varepsilon F(\varepsilon, x, y, \nabla w, \nabla^2 w)$ about w. Define

$$\theta = |G(w)|_{L^\infty(D)} \quad \text{and} \quad L_\theta(w)\rho = L(w)\rho + \theta\chi_1\rho_{yy}.$$

By Lemma 0.1, $L_\theta(w)$ is a degenerate elliptic operator. In terms of w,

$$a_{ij} = \varepsilon\frac{\partial F}{\partial w_{ij}}, \qquad a_i = \varepsilon\frac{\partial F}{\partial w_i}.$$

Hence

$$|a_{ij}|_{C^2} \quad \text{and} \quad |a_i|_{C^2} \leqslant C\varepsilon|w|_{C^4} \leqslant C_0\varepsilon|w|_{H^6}.$$

Therefore we have the following.

Corollary 1.4. *Suppose $\|w\|_{H^6} \leqslant 1$ and ε, θ, are sufficiently small, then there exists an integer $S_0(\varepsilon, \theta)$ depending on ε and θ such that if $g \in H^s$, $0 \leqslant s \leqslant s_0$, then there exists a unique solution $\rho \in H^s$ of the equation*:

$$\begin{aligned} &L_\theta(w)\rho = g \quad \text{in } D; \\ &\rho(x_0, y) = \rho(-x_0, y) = 0; \end{aligned} \tag{1.20}$$

and furthermore the following estimates are true

$$\|\rho\|_{H^s} \leqslant C_s\{\|g\|_{H^s} + \|w\|_{H^{s+4}}\|\rho\|_{H^2}\}. \tag{1.21}$$

Proof. Let L_θ^ν be the regularization (1.4) and ρ^ν be the unique H^s-solution. In terms of w, we have

$$\Gamma(s) \leqslant \varepsilon C_s^1(\|w\|_{H^{s+4}} + 1).$$

For $s \geqslant 2$, Theorem (1.1) implies

$$\|\rho^\nu\|_{H^s} \leqslant C_s\{\|g\|_{H^s} + \|w\|_{H^{s+4}}\|\rho^\nu\|_2\},$$

where C_s is independent of ν. Taking $\nu \to 0$, we have the estimate (1.21). For $s = 0$, (1.8) implies

$$\int u^2 \leqslant -(L_\theta u, u) = -(e^{\lambda x^2}g, u),$$

where $u = e^{\lambda x^2}g$. Therefore $\int u^2 \leqslant C\int g^2$. For $s = 1$, differentiate (1.20) by $\partial/\partial y$. Using integration by parts,

$$\left|\left(\frac{\partial u}{\partial y}, \left[L_\theta, \frac{\partial}{\partial y}\right]u\right)\right| \leqslant C(|a_{ij}|_{C^2} + |a_i|_{C^2})\|u\|_{H^1}^2.$$

By assumption $\|w\|_{H^6} \leqslant 1$, we have

$$\left|\left(\frac{\partial u}{\partial y}, \left[L_\theta, \frac{\partial}{\partial y}\right]u\right)\right| \leqslant C_1\varepsilon\|u\|_{H^1}^2.$$

By (1.8), we have

$$\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 \leqslant C_2\left\{\left\|\frac{\partial g}{\partial y}\right\|_{L^2}\left\|\frac{\partial u}{\partial y}\right\|_{L^2} + \varepsilon\|u\|_{H^1}^2\right\}.$$

It is equivalent to

$$\left\|\frac{\partial \rho}{\partial y}\right\|_{L^2}^2 \leqslant C_2'\left\{\left\|\frac{\partial g}{\partial y}\right\|_{L^2}\left\|\frac{\partial \rho}{\partial y}\right\|_{L^2} + \varepsilon\|\rho\|_{H^1}^2\right\}.$$

We have

$$(1 + a_{11})\rho_{xx} = g - \sum_{(i,j)\neq(1.1)} a_{ij}\rho_{x_i x_j} - \sum_i a_i \rho_{x_i}.$$

Multiplying ρ and integrating both sides,

$$\int \rho_x^2 \leqslant C_2\left\{\|\rho\|_{L^2}\|g\|_{L^2} + \varepsilon\|\rho\|_{H^1}^2\right\}.$$

Combining this and the above estimate of $\|\partial\rho/\partial y\|_{L^2}$, we have $\|\rho\|_{H^1} \leqslant C\|g\|_{H^1}$, provided ε and θ are small.

2.

In this section, we will modify the Nash-Moser-Hörmander's scheme to solve the nonlinear equation:

$$\text{(2.1)} \qquad \begin{aligned} &w_{xx} + \varepsilon F(\varepsilon, x, y, \nabla w, \nabla^2 w) = 0 \quad \text{in } D;\\ &w(x_0, y) = w(-x_0, y) = 0. \end{aligned}$$

Smoothing operators S_θ. We have a family of smoothing operator S_θ, $\theta > 1$, satisfying the following properties:

(S_1) S_θ: $H^s(D) \to H^{s'}(D)$ is a linear bounded operator for any s, s'.

(S_2) $\|S_\theta u\|_{H^s} \leqslant C_s\theta^{s-s'}\|u\|_{H^{s'}}$, if $s \geqslant s'$.

(S_3) $\|u - S_\theta u\|_{H^{s'}} \leqslant C_s\theta^{s'-s}\|u\|_{H^s}$, if $s \geqslant s'$.

One way to obtain the smoothing operators is the following: Consider a smooth domain $\tilde{D} \supset D$. We can extend functions u in $H^s(D)$ to a function $\tilde{u}$ of $\mathring{H}^s(\tilde{D})$, and satisfies

$$\|\tilde{u}\|_{H^s(\tilde{D})} \leqslant C_s\|u\|_{H^s(D)}.$$

Suppose $\tilde{S}_\theta$ be a family of smoothing operator in $\mathring{H}^s(\tilde{D})$ satisfying (S_1)–(S_3). Then we define S_θ in $H^s(D)$ by $S_\theta u = \tilde{S}_\theta\tilde{u}|_D$. It is easy to prove S_θ satisfies (S_1)–(S_3).

Nash-Moser-Hörmander's scheme. Choose $\mu_n = 2^n$, $S_n = S_{\mu_n}$, and $w_0 = 0$. We will construct w_n by induction on n as follows: Suppose $w_0, w_1, \cdots, w_n$ have been chosen. Define $w_{n+1} = w_n + \rho_n$ where ρ_n is the solution of

$$\text{(2.1)}\qquad \begin{aligned} &L_{\theta_n}(v_n)\rho_n = g_n \quad \text{in } D,\\ &\rho_n(x_0, y) = \rho_n(-x_0, y) = 0, \end{aligned}$$

where v_n is defined as $v_n = S_{\mu_n} w_n$,

$$\text{(2.2)}\qquad \theta_n = |G(v_n)|_{L^\infty},$$

and g_n will be specified later. For $j \leqslant n$, the quadratic error Q_j is defined as:

$$\begin{aligned} G(w_{j+1}) &= G(w_j) + L(w_j)\rho_j + Q_j(w_j, \rho_j)\\ &= G(w_j) + L_{\theta_j}(w_j)\rho_j - \theta_j\chi_1(\rho_j)_{yy} + Q_j(w_j, \rho_j)\\ &= G(w_j) + L_{\theta_j}(v_j)\rho_j + \big(L_{\theta_j}(w_j) - L_{\theta_j}(v_j)\big)\rho_j - \theta_j\chi_1(\chi_j)_{yy} + Q_j(w_j, \rho_j). \end{aligned}$$

Denote

$$\text{(2.3)}\qquad e_j = \big(L_{\theta_j}(w_j) - L_{\theta_j}(v_j)\big)\rho_j - \theta_j\chi_1(\rho_j)_{yy} + Q_j(w_j, \rho_j),$$

$$\text{(2.4)}\qquad E_j = \sum_{i=0}^{j-1} e_i.$$

Hence $G(w_{j+1}) = G(w_j) + g_j + e_j$. If we set $g_0 = -S_0G(w_0)$ and

$$g_j = S_{j-1}E_{j-1} - S_jE_j + (S_{j-1} - S_j)G(w_0) \quad \text{for } j > 0,$$

then

$$\text{(2.5)}\qquad \begin{aligned} G(w_{n+1}) &= G(w_0) + \sum_{j=0}^{n} g_j + E_n + e_n\\ &= G(w_0) - S_nG(w_0) - S_nE_n + E_n + e_n\\ &= (I - S_n)G(w_0) + (I - S_n)E_n + e_n. \end{aligned}$$

Theorem 2.1. *Suppose $F \in C^{s_*}$, $s_* > 6$, and ε is sufficiently small. Then the sequence $\{w_n\}$ converges to a solution w of* (0.4) *in H^{s_*-1}.*

In the following, we will give a proof of convergence of w_n. The proof is essentially the same as the usual proof of Nash-Moser-Hörmander's scheme. We include it for convenience. We will use the notation $\|u\|_s$ to denote Sobolev norm $\|u\|_{H^s}$.

First, recall a well-known lemma.

Lemma 2.2. *For any two functions u, v, the following inequality is true*:

$$\|D^\alpha u D^\beta v\|_{L^2} \leqslant C_s\{\|u\|_{L^\infty}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{L^\infty}\},$$

where

$$D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad s = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n.$$

This inequality follows from interpolational inequality immediately.

Fix an integer $\tilde{s} > 0$, and ε is chosen sufficiently small so that estimate (1.21) can be applied for $0 \leqslant s \leqslant \tilde{s}$. $0 < \bar{\varepsilon} < 1$, $b > 0$ are fixed. b will be chosen as large as possible. We want to find constant $C_1, C_2, \cdots, C_6$, and δ which depends only on $\tilde{s}$, and $\bar{\varepsilon}$, and independent of j, such that the following inequalities are true:

$$(\mathrm{P}_1)_j \qquad \|\rho_{j-1}\|_s \leqslant \delta \mu_{j-1}^{s-b} \quad \text{for } 0 \leqslant s \leqslant \tilde{s};$$

$$(\mathrm{P2})_j \qquad \|w_j\|_s \leqslant \begin{cases} C_1\delta & \text{if } s - b \leqslant -\bar{\varepsilon}, \\ C_1 \delta \mu_j^{s-b} & \text{if } s - b \geqslant \bar{\varepsilon}; \end{cases}$$

$$(\mathrm{P3})_j \qquad \|w_j\|_6 \text{ and } \|v_j\|_6 \leqslant 1;$$

$$(\mathrm{P4})_j \qquad \|w_j - v_j\|_s \leqslant C_2 \delta \mu_j^{s-b} \quad \text{for } 0 \leqslant s \leqslant \tilde{s};$$

$$(\mathrm{P5})_j \qquad \|v_j\|_s \leqslant \begin{cases} C_3\delta & \text{if } s - b \leqslant -\bar{\varepsilon}, \\ C_3 \delta \mu_j^{s-b} & \text{if } s - b \geqslant \bar{\varepsilon}; \end{cases}$$

$$(\mathrm{P6})_j \qquad \|e_{j-1}\|_s \leqslant C_4 \delta^2 \mu_{j-1}^{s-b} \quad \text{for } 0 \leqslant s \leqslant \tilde{s} - 2;$$

$$(\mathrm{P7})_j \qquad \|g_j\|_s \leqslant C_5 \delta^2 \mu_j^{s-b} \quad \text{for } 0 \leqslant s \leqslant \tilde{s};$$

$$(\mathrm{P8})_j \qquad \theta_j \leqslant C_6 \delta \mu_j^{4-b}.$$

We will prove $(\mathrm{P1})_j$–$(\mathrm{P8})_j$ by induction on j. At the beginning, we may assume $G(w_0) \in H^{s_*}$ and ε is very small so that $(\mathrm{P1})_0$–$(\mathrm{P8})_0$ is true. For $j = 0$, we only have to check $(\mathrm{P7})_0$ and $(\mathrm{P8})_0$. Now suppose $(\mathrm{P1})_j$–$(\mathrm{P8})_j$ are true for $0 \leqslant j \leqslant n$, and we want to prove $(\mathrm{P1})_{n+1}$–$(\mathrm{P8})_{n+1}$.

$(\mathrm{P1})_{n+1}$: Applying Corollary 1.4, we have for $0 \leqslant s \leqslant \tilde{s}$,

$$\begin{aligned} \|\rho_n\|_s &\leqslant C_s\{\|g_n\|_s + \|v_n\|_{s+4}\|g_n\|_2\} \\ &\leqslant C_s\{C_5\delta^2\mu_n^{s-b} + C_3C_5\delta^2\mu_n^{s+4-b}\mu_n^{2-b}\} \qquad (2.6) \\ &\leqslant C_s(C_5 + C_3C_5)\delta^2\mu_n^{s-b}, \end{aligned}$$

provided $6 \leqslant b$. Hence, if δ is small, $\|\rho_n\|_s \leqslant \delta\mu_n^{s-b}$.

$(\mathrm{P2})_{n+1}$: $w_{n+1} = w_n + \rho_n = \sum_{j=0}^n \rho_j$,

$$\|w_{n+1}\|_s \leqslant \sum_{j=0}^{n} \|\rho_j\|_s \leqslant \delta \sum_{j=0}^{n} \mu_j^{s-b}.$$

If $s-b \leqslant -\bar{\varepsilon}$, $\|w_{n+1}\|_s \leqslant \delta\sum_{j=0}^{\infty}\mu_j^{-\bar{\varepsilon}} = C_1\delta$; if $s-b \geqslant \bar{\varepsilon}$,

$$\|w_{n+1}\|_s \leqslant \delta\mu_{n+1}^{s-b}\sum_{j=0}^{n}\left(\frac{\mu_i}{\mu_{n+1}}\right)^{s-b}$$

$$\leqslant \delta\mu_{n+1}^{s-b}\sum_{j=0}^{\infty}(2^{-j})^{\bar{\varepsilon}} = C_1\delta\mu_{n+1}^{s-b}.$$

$(\mathrm{P3})_{n+1}$: $\|w_{n+1}\|_6 \leqslant C_1\delta$ by (2.6) and $(\mathrm{P2})_{n+1}$,

$$\|v_{n+1}\|_6 \leqslant \tilde{C}\|w_{n+1}\|_6 \leqslant C_1\tilde{C}\delta,$$

so if δ is chosen very small, then

$$\|w_{n+1}\|_6 \leqslant 1 \quad \text{and} \quad \|v_{n+1}\|_6 \leqslant 1.$$

$(\mathrm{P4})_{n+1}$: For $0 \leqslant s \leqslant \tilde{s}$,

$$\|w_{n+1} - v_{n+1}\|_s = \|w_{n+1} - S_{\mu_{n+1}}w_{n+1}\|_s \leqslant C_s\mu_{n+1}^{s-\tilde{s}}\|w_{n+1}\|_{\tilde{s}}$$

$$\leqslant C_sC_1\delta\mu_{n+1}^{s-\tilde{s}}\mu_{n+1}^{\tilde{s}-b} \equiv C_2\delta\mu_{n+1}^{s-b}.$$

$(\mathrm{P5})_{n+1}$: $\|v_{n+1}\|_{\tilde{s}+4} \leqslant C_s\mu_{n+1}^4\|w_{n+1}\|_{\tilde{s}} \leqslant C_sC_1\delta\mu_{n+1}^{\tilde{s}+4-b}$,

$$\|v_{n+1}\|_{b+\bar{\varepsilon}} \leqslant \|v_{n+1} - w_{n+1}\|_{b+\bar{\varepsilon}} + \|w_{n+1}\|_{b+\bar{\varepsilon}}$$

$$\leqslant (C_2\delta + C_1\delta)\mu_{n+1}^{\bar{\varepsilon}}.$$

Using interpolational inequality for $b+\bar{\varepsilon} \leqslant s \leqslant \tilde{s}+4$,

$$\|v_{n+1}\|_s \leqslant C_3\delta\mu_{n+1}^{s-b}.$$

For $0 \leqslant s \leqslant b-\bar{\varepsilon}$, we have

$$\|v_{n+1}\|_s \leqslant C_s\|w_{n+1}\|_s \leqslant C_1C_s\delta.$$

$(\mathrm{P6})_{n+1}$: $e_n = (L_{\theta_n}(w_n) - L_{\theta_n}(v_n))\rho - \theta_n\chi_1(\rho_n)_{yy} + Q_n(w_n, \rho_n)$

$$\equiv e_n' + e_n'' + e_n''',$$

$$e_n' = \big(L_{\theta_n}(w_n) - L_{\theta_n}(v_n)\big)\rho_n.$$

Using Lemma 2.1, we have

$$\|e_n'\|_0 \leqslant C\|w_n - v_n\|_3\|\rho_n\|_3 \leqslant C_2\delta^2\mu_n^{3-b}\mu_{n-1}^{3-b}$$

$$= C_2\delta^2\left(\frac{1}{2}\right)^{3-b}\mu_n^{6-2b} = \left(2^{b-3}C_2\delta^2\right)\mu_n^{6-b}\mu_n^{-b} \leqslant 2^{b-3}C_2\delta^2\mu_n^{-b},$$

and

$$\|e_n'\|_{\tilde{s}-2} \leqslant C\{\|w_n - v_n\|_{\tilde{s}}\|\rho_n\|_{C^2} + \|w_n - v_n\|_{C^2}\|\rho_n\|_{\tilde{s}}\}$$
$$\leqslant C\{\|w_n - v_n\|_{\tilde{s}}\|\rho_n\|_4 + \|w_n - v_n\|_4\|\rho_n\|_{\tilde{s}}\}$$
$$\leqslant C\{C_2\delta^2\mu_n^{\tilde{s}-b}\mu_n^{4-b} + C_2\delta^2\mu_n^{4-b}\mu_n^{\tilde{s}-b}\}$$
$$\leqslant 2CC_2\delta^2\mu_n^{(\tilde{s}-2)-b}\mu_n^{6-b} \leqslant 2CC_2\delta^2\mu_n^{(\tilde{s}-2)-b}$$

by (2.6). Then, using interpolational inequality, we have, $0 \leqslant s \leqslant \tilde{s} - 2$, $\|e_n'\|_s \leqslant C_s\delta^2\mu_n^{s-b}$ for some constant C_s. Thus

$$\|e_n''\|_s \leqslant C_s\theta_n\|\rho_n\|_{s+2} \leqslant C_sC_6\delta^2\mu_n^{4-b}\mu_n^{s-b+2} \leqslant C_sC_6\delta^2\mu_n^{s-b},$$

here we use $(P8)_n$. Since

$$e_n''' = G(w_{n+1}) - G(w_n) - L(w_n)\rho_n = \int_0^1 (1-t)\frac{\partial^2}{\partial t^2}G(w_n + t\rho)\,dt,$$

using Lemma 2.1, we have

$$\|e_n'''\|_0 \leqslant C\big(\|w_n\|_{C^2(D)} + \|w_{n+1}\|_{C^2(D)}\big)\|\rho_n\|_3^2 \leqslant \tilde{C}\delta^2\mu_n^{2(3-b)} \leqslant \tilde{C}\delta^2\mu_n^{-6},$$

here we use $(P3)_n$, $(P3)_{n+1}$, and $(P1)_{n+1}$. Similarly,

$$\|e_n'''\|_{\tilde{s}-2} \leqslant C\Big\{\big(\|w_{n+1}\|_{\tilde{s}} + \|w_n\|_{\tilde{s}}\big)\|\rho_n\|_4^2 + \big(\|w_{n+1}\|_4 + \|w_n\|_4\big)\|\rho_n\|_4\|\rho_n\|_{\tilde{s}}\Big\}$$
$$\leqslant C\{2C_1\delta^3\mu_{n+1}^{\tilde{s}-b}\mu_n^{2(4-b)} + \delta^2\mu_n^{\tilde{s}-b+4-b}\} \leqslant \tilde{C}_1\delta^2\mu_n^{\tilde{s}-2-b}.$$

By interpolational inequality, we have, for $0 \leqslant s \leqslant \tilde{s} - 2$,

$$\|e_n'''\|_s \leqslant \tilde{\tilde{C}}_1\delta^2\mu_n^{s-b}.$$

Combining estimates of $\|e_n'\|_s$, $\|e_n''\|_s$, $\|e_n'''\|_s$, we have proved $(P6)_{n+1}$.

$(P7)_{n+1}$: $$g_{n+1} = S_nE_n - S_{n+1}E_{n+1} + (S_n - S_{n+1})G(w_0)$$
$$= (S_n - S_{n+1})E_n - S_{n+1}e_n + (S_n - S_{n+1})G(w_0);$$
$$E_n = \sum_{j=0}^{n-1} e_j;$$

(2.7) $$\|E_n\|_{\tilde{s}-2} \leqslant \sum_{j=0}^{n-1}\|e_j\|_{\tilde{s}-2} \leqslant C_4\delta^2\sum_{j=0}^{n-1}\mu_j^{s-2-b} \leqslant C_4\delta^2\mu_n^{\tilde{s}-2-b},$$

provided $\tilde{s} - 2 - b > 0$;

$$\|g_{n+1}\|_0 \leqslant C_s\{\mu_n^{2-\tilde{s}}\|E_n\|_{\tilde{s}-2} + \|e_n\|_0 + \mu_n^{-s_*}\|G(w_0)\|_{s_*}\} \leqslant C_4'\delta^2\mu_{n+1}^{-b};$$

provided ε is sufficiently small and

(2.8) $$s_* \geqslant b;$$

$$\|g_{n+1}\|_{\tilde{s}} \leqslant C\{\mu_{n+1}^2\|E_n\|_{\tilde{s}-2} + \mu_{n+1}^2\|e_n\|_{\tilde{s}-2} + \mu_{n+1}^{\tilde{s}-s_*}\|G(w_0)\|_{s_*}\} \leqslant C_4''\delta^2\mu_{n+1}^{\tilde{s}-b}.$$

By interpolational inequality, we have proved $(\mathrm{P7})_{n+1}$.

$(\mathrm{P8})_n$: $\theta_{n+1} = \|G(v_{n+1})\|_{L^\infty}$. By (2.5),

$$\begin{aligned} G(v_{n+1}) &= G(w_{n+1}) + G(v_{n+1}) - G(w_{n+1}) \\ &= (I - S_n)G(w_0) + (I - S_n)E_n + e_n + G(v_{n+1}) - G(w_{n+1}). \end{aligned}$$

$$\begin{aligned} \|G(v_{n+1})\|_{L^\infty} &\leqslant C\big\{\|(I - S_n)G(w_0)\|_2 \\ &\quad + \|(I - S_n)E_n\|_2 + \|e_n\|_2 + \|v_{n+1} - w_{n+1}\|_4\big\} \\ &\leqslant C\big\{\mu_n^{2-s_*}\|G(w_0)\|_{s_*} + \mu_n^{4-\bar{s}}\|E_n\|_{\bar{s}-2} + \mu_n^{2-b} + \|v_{n+1} - w_{n+1}\|_4\big\} \\ &\leqslant C\big\{\mu_n^{2-s_*}\|G(w_0)\|_{s_*} + C_4'\delta^2\mu_n^{\bar{s}-2-b+4-\bar{s}} + C_2\delta^2\mu_{n+1}^{4-b}\big\} \leqslant C_6\delta\mu_{n+1}^{4-b}. \end{aligned}$$

Hence, if we assume (2.6), (2.7), and (2.8), we have proved the induction step.

Proof of Theorem 2.1. Suppose $s_* > 6$. Choose $b = s_* - 1/2$, $\bar{\varepsilon} = 1/2$. For $n \geqslant m$, $s < b$,

$$\|w_n - w_m\|_s \leqslant \sum_{j=m}^{n} \|\rho_j\|_s \leqslant \delta \sum_{j=m-1}^{n-1} (2^{-j})^{b-s} < +\infty.$$

Hence w_n converges to w in H^{s_*-1}. By (2.15),

$$G(w_{n+1}) = (I - S_n)G(w_0) + (I - S_n)E_n + e_n.$$

By $(\mathrm{P6})_j$, we have $\lim_{n\to+\infty}\|G(w_{n+1})\|_{s_*-1} = 0$. Hence $G(w) = 0$, i.e., we have found a solution of (0.4).

Remark. Suppose our original metric is C^s. Then

$$F(\varepsilon, x, y, \nabla w, \nabla^2 w) \in C^{s-3}.$$

By Theorem 2.1, we require $s - 3 > 6$, i.e., $s > 9$ and the solution $w \in H^{s-4} \subset C^{s-6}$.

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